# A plane symmetric universe filled with disordered radiation



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# A plane symmetric universe filled with disordered radiation

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**Abstract.** A plane symmetric model of the universe has been derived which will include all plane symmetric space–times filled with disordered radiation. Some properties of the model have been studied in some detail.

## 1. Introduction

In our previous paper (Roy and Singh 1976), we derived a cosmological model which evolved from an earlier stage when $t=t_1$. In this model, radiation was in collisional equilibrium where the equation of state assumes the form $\rho=3p$. It is therefore worthwhile to obtain models using this equation of state. At $t=t_2$, it changes over to another model in which matter becomes tenuous so that $p$ is zero. A model has been given by Heckmann and Schucking (1962) for which $p$ is zero. In this paper, a plane symmetric model has been derived which will include all plane symmetric space–times filled with disordered radiation. Some properties of the model have been studied in some detail.

## 2. Derivation of the line element

The cylindrically symmetric metric is considered in the form given by Marder (1958)

$$ds^2=A^2(dt^2-dx^2)-B^2\,dy^2-C^2\,dz^2, \tag{2.1}$$

where $A$, $B$, $C$ are functions of $t$ only.

The energy–momentum tensor for a distribution of disordered radiation can be regarded as a perfect fluid

$$T_{ij}=(\rho+p)\lambda_i\lambda_j-pg_{ij}, \tag{2.2}$$

together with the equation of state

$$\rho=3p. \tag{2.3}$$

The coordinates are assumed to be comoving so that $\lambda^1=\lambda^2=\lambda^3=0$.

The field equations

$$R_{ij}-\tfrac{1}{2}Rg_{ij}=-8\pi T_{ij} \qquad (\text{with } C=G=1)$$

for the line element (2.1) are as follows:

$$\frac{1}{A^2}\left[\left(\frac{B_{44}}{B}+\frac{C_{44}}{C}-\frac{A_4}{A}\left(\frac{B_4}{B}+\frac{C_4}{C}\right)+\frac{B_4C_4}{BC}=-8\pi p,\right.\right. \tag{2.4}$$

$$\frac{1}{A^2}\left[\left(\frac{A_4}{A}\right)_4+\frac{C_{44}}{C}\right]=-8\pi p, \tag{2.5}$$

$$\frac{1}{A^2}\left[\left(\frac{A_4}{A}\right)_4+\frac{B_{44}}{B}\right]=-8\pi p, \tag{2.6}$$

$$\frac{1}{A^2}\left[\frac{A_4}{A}\left(\frac{B_4}{B}+\frac{C_4}{C}\right)+\frac{B_4C_4}{BC}\right]=24\pi p. \tag{2.7}$$

From equations (2.4)–(2.7), we have

$$\left(\frac{A_4}{A}\right)_4+\frac{B_{44}}{B}+\frac{C_{44}}{C}+\frac{B_4C_4}{BC}=0; \tag{2.8}$$

from equations (2.5) and (2.6), we have

$$\frac{B_{44}}{B}=\frac{C_{44}}{C} \tag{2.9}$$

and from equations (2.4) and (2.5), we have

$$\left(\frac{A_4}{A}\right)_4+\frac{A_4}{A}\left(\frac{B_4}{B}+\frac{C_4}{C}\right)=\frac{B_{44}}{B}+\frac{B_4C_4}{BC}. \tag{2.10}$$

Let $BC=\nu$ and $B/C=\mu$; then from equation (2.9) we have

$$\frac{\mu_4}{\mu}=\frac{K}{\nu}, \tag{2.11}$$

where $K$ is a constant of integration.

From equation (2.10), we have

$$BC\left(\frac{A_4}{A}\right)_4+\frac{A_4}{A}(BC)_4=CB_{44}+B_4C_4,$$

which gives

$$BC\frac{A_4}{A}=B_4C+L,$$

where $L$ is a constant of integration.

Thus,

$$\frac{A_4}{A}=\frac{M}{\nu}+\frac{\nu_4}{2\nu}, \tag{2.12}$$

where $M=L+\frac{1}{2}K$.

From equations (2.8), (2.10) and (2.12), we have

$$M\frac{\nu_4}{\nu^2}+\frac{3}{4}\frac{\nu_4^2}{\nu^2}-\frac{3}{2}\frac{\nu_{44}}{\nu}-\frac{K^2}{4\nu^2}=0. \tag{2.13}$$

Let $\nu_4 = \tau(\nu)$, then $\nu_{44} = \tau'\tau$ where the prime denotes differentiation with respect to $\nu$. From equation (2.13), we get

$$\int \frac{2\tau\,\mathrm{d}\tau}{\tau^2 + \frac{4}{3}M\tau - \frac{1}{3}K^2}$$

$$= \ln\left[(\tau^2 + \tfrac{4}{3}M\tau - \tfrac{1}{3}K^2)\left(\frac{3\tau + 2M - (4M^2+3K^2)^{1/2}}{3\tau + 2M + (4M^2+3K^2)^{1/2}}\right)\right]$$

$$= \ln q\nu,$$

where $q$ is a constant.

Therefore,

$$\nu = \frac{1}{q}(\tau^2 + \tfrac{4}{3}M\tau - \tfrac{1}{3}K^2)\left(\frac{3\tau + 2M + (4M^2+3K^2)^{1/2}}{3\tau + 2M - (4M^2+3K^2)^{1/2}}\right)^{2M(4M^2+3K^2)^{-1/2}} \tag{2.14}$$

Let

$$\tau^2 + \tfrac{4}{3}M\tau - \tfrac{1}{3}K^2 = (\tau+\alpha)(\tau-\beta),$$

where $\alpha = \frac{2}{3}M + \frac{1}{3}(4M^2+3K^2)^{1/2}$ and $\beta = -\frac{2}{3}M + \frac{1}{3}(4M^2+3K^2)^{1/2}$. Hence

$$\nu = \frac{1}{q}(\tau+\alpha)^{2\alpha/(\alpha+\beta)}(\tau-\beta)^{2\beta/(\alpha+\beta)}. \tag{2.15}$$

Since $\nu_4 = \tau(\nu)$:

$$\int \frac{\mathrm{d}\nu}{\tau(\nu)} = t + t' \tag{2.16}$$

where $t'$ is a constant of integration.

From equation (2.15)

$$\mathrm{d}\nu = \frac{2\nu}{\alpha+\beta}\left(\frac{\alpha}{\tau+\alpha} + \frac{\beta}{\tau-\beta}\right)\mathrm{d}\tau$$

or

$$t + t' = \frac{2}{q}\int \left(\frac{\tau+\alpha}{\tau-\beta}\right)^{(\alpha-\beta)/(\alpha+\beta)} \mathrm{d}\tau. \tag{2.17}$$

From equation (2.11):

$$\nu \frac{\mathrm{d}\mu}{\mathrm{d}\nu}\nu_4 = K\mu,$$

or

$$\nu \frac{\mathrm{d}\mu}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} = K\mu,$$

which may be written

$$\frac{1}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}\tau}\frac{q}{2}\left(\frac{\tau-\beta}{\tau+\alpha}\right)^{(\alpha-\beta)/(\alpha+\beta)}=\frac{K}{\nu},$$

or

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(\ln\mu)=\frac{2K}{q\nu}\left(\frac{\tau+\alpha}{\tau-\beta}\right)^{(\alpha-\beta)/(\alpha+\beta)}=2K(\tau+\alpha)^{-1}(\tau+\beta)^{-1}.$$

Therefore,

$$\mu=\left(\frac{\tau-\beta}{\tau+\alpha}\right)^{2K/(\alpha+\beta)} \tag{2.18}$$

From equations (2.11) and (2.12), we have

$$\frac{A_4}{A}=\frac{M}{K}\frac{\mu_4}{\mu}+\frac{\nu_4}{2\nu}.$$

Therefore

$$A=\gamma\mu^{M/K}\nu^{1/2}, \tag{2.19}$$

where $\gamma$ is a constant.

The metric (2.1) then becomes:

$$\begin{aligned}\mathrm{d}s^2=&(4\gamma^2/q_3)(\tau+\alpha)^{[4(\alpha-M)-2\beta]/(\alpha+\beta)}(\tau-\beta)^{[4(\beta+M)-2\alpha]/(\alpha+\beta)}\mathrm{d}\tau^2\\&-(\tau+\alpha)^{(2\alpha-4M)/(\alpha+\beta)}(\tau-\beta)^{(2\beta+4M)/(\alpha+\beta)}\,\mathrm{d}X^2\\&-(\tau+\alpha)^{2(\alpha-K)/(\alpha+\beta)}(\tau-\beta)^{2(\beta+K)/(\alpha+\beta)}\,\mathrm{d}Y^2\\&-(\tau+\alpha)^{2(\alpha+K)/(\alpha+\beta)}(\tau-\beta)^{2(\beta-K)/(\alpha+\beta)}\,\mathrm{d}Z^2\end{aligned} \tag{2.20}$$

where $\tau$ is a new time coordinate.

By introducing a new coordinate and constants:

$$\begin{aligned}&t=\frac{\tau+\alpha}{\tau-\beta}, \qquad \alpha'=\frac{\alpha}{\alpha+\beta}, \qquad \beta'=\frac{\beta}{\alpha+\beta}\\&M'=\frac{M}{\alpha+\beta}, \qquad K'=\frac{K}{\alpha+\beta}, \qquad L=\frac{4\gamma^2}{q_3}(\alpha+\beta)^4,\end{aligned} \tag{2.21}$$

$$(\alpha+\beta)^{(\alpha'+\beta')}X=x, \qquad (\alpha+\beta)^{(\alpha'+\beta')}Y=y, \qquad (\alpha+\beta)^{(\alpha'+\beta')}Z=z,$$

the above metric (2.20) reduces to the form

$$\begin{aligned}\mathrm{d}s^2=&L(t)^{4(\alpha'-M')-2\beta'}(t-1)^{-6}\,\mathrm{d}t^2-(t)^{2(\alpha'-2M')}(t-1)^{-2}\,\mathrm{d}x^2-(t)^{2(\alpha'-K')}(t-1)^{-2}\,\mathrm{d}y^2\\&-(t)^{2(\alpha'+K')}(t-1)^{-2}\,\mathrm{d}z^2.\end{aligned} \tag{2.22}$$

### *2.1. A particular case*

Let $M=0$. Then equation (2.13) becomes

$$3(\nu^{1/2})_{44}+\tfrac{1}{4}K^2\nu^{-3/2}=0. \tag{2.23}$$

Solution of the above differential equation is given by

$$\nu = \tfrac{3}{4}[\tfrac{1}{12}K^2 - \tfrac{1}{9}N^2(t+t_0)^2], \tag{2.24}$$

where $K$, $N$ and $t_0$ are constants of integration.

Density of the distribution is given by

$$8\pi\rho = (4\alpha^2\gamma^3)^{-1}[\tfrac{4}{3}N^2(t+t_0)^2 - K^2], \tag{2.25}$$

which is negative and hence the above solution is not physically plausible.

## 3. Some physical and geometrical features of equation (2.22)

The density in the model is given by

$$8\pi\rho = L^{-1}(t)^{2\beta'-4(\alpha'-M')}[(3\alpha'^2 - 4M'\alpha' - K'^2)t^{-2} + 3\beta'^2 + 4M'\beta' - K'^2]. \tag{3.1}$$

The reality condition $\rho > 0$ leads to

$$\frac{1}{t^2} > \frac{K'^2 - 4M'\beta' - 3\beta'^2}{3\alpha'^2 - 4M'\alpha' - K'^2}. \tag{3.2}$$

The flow vector $\lambda^i$ of the distribution is given by

$$\begin{aligned} &\lambda^1 = \lambda^2 = \lambda^3 = \lambda_1 = \lambda_2 = \lambda_3 = 0. \\ &\lambda^4 = L^{-1/2}(t-1)(t)^{\beta'-2(\alpha'-M')} \\ &\lambda_4 = L^{1/2}(t-1)^{-1}(t)^{2(\alpha'-M')-\beta'}. \end{aligned} \tag{3.3}$$

The flow vector $\lambda^i$ satisfies the equation of the geodesics $\lambda^i_{;j}\lambda^j = 0$. The semicolon represents covariant differentiation. The scalar of expansion is given by

$$\Theta = L^{-1/2}\{[\beta' - 2(\alpha' - M')](t)^{\beta'-2(\alpha'-M')-1}(t-1)^2 + [\alpha' - 2(\beta' + M')](t)^{\beta'-2(\alpha'+M')}(t-1)^2\}. \tag{3.4}$$

The tensor of rotation $W_{ij}$ is zero. Thus the fluid filling the universe is irrotational.

The components of the shear tensor $\sigma_{ij}$ are:

$$\begin{aligned} &\sigma_{11} = L^{-1/2}\{[\beta' - 2(\alpha' - M')](t)^{\beta'-2M'-1} + [\alpha' - 2(\beta' + M')](t)^{\beta'-2M'}] \\ &\sigma_{22} = L^{-1/2}\{[\beta' - 2(\alpha' - M')](t)^{\beta'+2(M'-K')-1} + [\alpha' - 2(\beta' + M')](t)^{\beta'+2(M'-K')}\} \\ &\sigma_{33} = L^{-1/2}\{[\beta' - 2(\alpha' - M')](t)^{\beta'+2(M'+K')-1} + [\alpha' - 2(\beta' + M')](t)^{\beta'+2(M'+K')}\}. \\ &\sigma_{44} = 0. \end{aligned} \tag{3.5}$$

The red-shift in the model is given by

$$\frac{\lambda + \delta\lambda}{\lambda} = L^{1/2}\frac{(t)^{2(\alpha'-M')-\beta'}(t-1)^{-3}[(t_1)^{2(\alpha'-K')}(t_1-1)^2 + U_z]}{(t)^{2(\alpha'-K')}(t-1)^2[L(t)^{4(\alpha'-M')-2\beta'}(t-1)^{-6} - U^2]}, \tag{3.6}$$

where $U$ is the velocity of the source at the time of emission and $U_z$ is the $z$ component of the velocity.

The metric (2.22) is not conformal to flat space–time. It is not a particular case of the Lemaitre universe.

## References

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